Optimal Transition to Backstop Substitutes for Nonrenewable Resources*

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Abstract: We analyze the optimal transition from a primary, nonrenewable resource to a backstop substitute for a class of problems characterized by the property that the backstop cost decreases continuously as learning from R&D efforts accumulates to increase the knowledge base. The transition policy consists of the R&D process and of the time profiles of the primary and backstop resource supply rates. We find that the optimal R&D process follows a Most Rapid Approach Path: if R&D is at all worthwhile, the associated knowledge process should approach some (endogenously derived) target process as rapidly as possible and proceed along it thereafter. Thus, R&D should be initiated without delay at the highest affordable rate and slow down later on. This pattern contrasts previous findings that typically recommend a single-humped R&D process with a possible initial delay.

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1. INTRODUCTION

The standard theory of a resource exploitation industry facing a backstop technology postulates that the resource will be abandoned when its cost reaches that of the backstop resource, at which time a transition to the backstop resource takes place at once (Heal, 1976, Dasgupta and Heal, 1979). A similar pattern holds when the arrival date of the backstop is uncertain, although the uncertainty may have substantial affects on the resource price and extraction profiles (Dasgupta and Heal, 1974, Dasgupta and Stiglitz, 1981). Kamien and Schwartz (1971, 1978), and Dasgupta, Heal and Majumdar (1977) incorporated endogenous R&D efforts that accumulate in the form of knowledge to affect the probability of developing a competitive backstop. Deshmukh and Pliska (1985) add exploration activities and synthesize the different models within a unified framework of analysis.

While the bulk of the literature obtains a once-and-for-all adoption of the backstop technology, some, notably Hoel (1978) and Hung and Quyen (1993), find that certain conditions call for a simultaneous use of the primary and backstop resources. Hung and Quyen (1993) extend Dasgupta and Stiglitz' (1981) framework by adding a decision regarding the time to initiate an R&D program, although the program itself (i.e., the schedule of R&D efforts) is exogenous. Assuming a fixed R&D expenditure, they investigate the effects of uncertainty regarding the technological breakthrough arrival date on the depletion policy. By allowing the marginal cost of producing the backstop to depend on the production rate, they find a gradual, rather than abrupt, transition to the backstop resource. The simultaneous exploitation of the primary and backstop resources can be optimal also in Hoel's (1978) framework, in which the focus is on the market structure of the backstop supply, if the substitute is supplied competitively (this, however, no longer holds when the monopolist controls both the primary and the backstop supplies). In a more empirically oriented study, Chakravorty et al. (1997) consider various scenarios of endogenous substitution among energy resources and find that simultaneous use is the most plausible transition mode to solar energy technologies.

A common feature in backstop R&D modeling is that the backstop technology arrival (or improvement) is a discrete event whose occurrence (which may be governed by uncertainty) is affected by the R&D policy. In this work we depart from this characteristic aspect by considering a continuous improvement of an existing backstop technology, manifested through R&D efforts that accumulate in the form of knowledge to reduce the cost of backstop supply. While some backstop technologies advance in discrete steps or via major breakthrough discoveries, examples of continuously improving backstops are not rare, including renewable energy technologies such as solar, wind, hydro or ocean thermal energy conversion (see data on photovoltaic electricity in Chakravorty et al. 1997 and references therein). Our analysis is developed for such cases.

Optimal R&D processes under the discrete-event framework typically follow a single-humped path—an increase followed by a decrease—with a possible delay in initiating the R&D program. Our model displays a different pattern: if R&D is at all worthwhile, it should be implemented immediately at a maximal affordable rate until the knowledge process attains some (endogenously derived) target process which depends, inter alia, on the relative costs of the primary and backstop resources. From this date on, R&D should be so tuned as to retain the knowledge process along the target process. Thus, the optimal R&D policy is to approach as rapidly as possible the target knowledge process and proceed along it forever; a behavior akin to Spence and Starrett's (1975) Most Rapid Approach Path (MRAP) policy. When the marginal cost of the primary resource depends on its rate of extraction, the transition is smooth and the rate of primary resource supply decreases continuously in time and approaches zero as the resource stock is nearing depletion.

The structure of the paper is as follows. In Section 2, we formulate a transition policy in terms of the primary and backstop resources supply rates and the R&D efforts. The optimal policy is characterized in Section 3. Section 4 concludes and the appendix contains the technical derivations.

2. FORMULATION OF A TRANSITION POLICY

The primary resource serves as input in the production of final goods and can be substituted with a backstop resource. Alternatively, the backstop technology might correspond to a production process that does away with the primary resource. While the stock of the primary resource is nonrenewable and finite, the backstop resource is practically limited only by its cost, which declines with technological progress resulting from R&D efforts. Realistic scenarios may involve several primary resources and several alternative backstop technologies. For simplicity, we aggregate these options into a single nonrenewable resource and a single backstop, ignoring important differences within each class of resources.

Demand: The instantaneous demand $D(p)$ for the resource is a decreasing function of the resource price p. The inverse demand, $D^{-1}(q)$, represents the price along the demand curve corresponding to any rate of supply q. The gross consumer surplus from the supply at the rate q is given by $G(q) = \int_0^q D^{-1}(z)dz$. We maintain stationary demand; an extension to demand that increases over time (say, with population growth) is discussed in Tsur and Zemel (1998) where it is shown that the trajectory of the optimal process is sensitive to this change, but the qualitative nature of the optimal policy is retained.

Supply of the nonrenewable (primary) resource: The resource cost is composed of extraction, or engineering, cost (including delivery, interest and depreciation on investment in facilities, wages and disposable equipment) and scarcity rent. Let $C(q^c)$ represent the engineering cost of supplying the primary resource at the rate q^c (the superscript/subscript c

stands for conventional). It is assumed that $C(0) = 0$ and that the marginal cost $M_c(q^c) \equiv dC(q^c)/dq^c$ increases with q^c. This is so because the last unit of the resource should be supplied from the cheapest source (plant, mining site etc.) that is still operating below capacity and the supply of larger quantities require the operation of the more expensive sources.

The scarcity rent will show up below in the formulation of the dynamic solution through the shadow price (the costate variable) associated with the resource stock X_t, which changes over time according to

$$\dot{X}_t \equiv dX_t / dt = -q_t^c \tag{1}$$

Backstop supply: The backstop technology improves as R&D activities are translated into knowledge via learning. This implies that the marginal cost M_s of backstop supply is a decreasing function of the state of knowledge K_t available at time t (subscript/superscript s stands for substitute or backstop resource). The latter, in turn, consists of all the R&D investments $\{R_\tau, \tau \leq t\}$ that had taken place up to time t. Assuming the backstop technology admits constant returns to scale, the cost of supplying the backstop resource at the rate q^s is specified as $M_s(K_t)q^s$. We acknowledge that restricting the marginal cost to depend on the knowledge state alone disregards possible cost determinants, such as dependence on the supply rate q^s or dependence on the cumulative supply of the backstop resource due to learning-by-doing (at least during earlier stages of market penetration). We note, however, that incorporating these factors complicates the analysis with very little effect on the results.

As time goes by, part of this knowledge may become obsolete due to aging, new discoveries or transition to new technologies that are not directly related to (and are not the result of) backstop research. Such technological progress can have a bearing on backstop production processes and it reduces the value of backstop knowledge in use prior to its arrival.

The balance between the rate of R&D investment, R_t, and the rate at which existing knowledge is lost determines the rate of knowledge accumulation

$$\dot{K}_t \equiv dK_t / dt = R_t - \delta K_t \tag{2}$$

where K is measured in monetary units and the constant δ is a knowledge depreciation parameter (the special case $\delta = 0$ entails no difficulty and will be discussed below). Equation (2) assumes the usual capital-investment relation, with knowledge as the capital and R&D the investment. R&D, then, increases the stock of knowledge, which in turn reduces the backstop cost in a nonlinear fashion via the function $M_s(K)$. Although knowledge accumulation is linear in R&D, the knowledge-backstop cost relation can assume a nonlinear form (see left panel of Figure 1).

Integrating (2), we find

$$K_t = \int_0^t R_\tau e^{\delta(\tau - t)} d\tau + K_0 e^{-\delta t} \tag{3}$$

It is maintained that the initial level K_0 is sufficiently low to warrant R&D investment (see Assumption 1 below).

Social benefit: The direct cost of supplying q^c+q^s is $C(q^c) + M_s(K_t)q^s$. The net consumer and producer surplus generated by $q = q^c+q^s$ is $G(q^c+q^s) - [C(q^c) + M_s(K_t)q^s]$, where the gross consumer surplus $G(q) = \int_0^q D^{-1}(z)dz$ is defined above. Adding the costs of R&D, the net social benefit at time t is

$$G(q_t^c + q_t^s) - C(q_t^c) - M_S(K_t)q_t^s - R_t . \tag{4}$$

The transition policy (resources use and R&D): A transition policy consists of three control (flow) and two state processes: The flow processes are q_t^c (primary resource supply), q_t^s (backstop resource supply) and R_t (R&D investment). The state processes are X_t

(remaining reserves of the nonrenewable resource), and K_t (knowledge). The transition policy $\Gamma = \{q_t^c, q_t^s, R_t, t \geq 0\}$ determines the evolution of the state processes X_t and K_t via (1)-(2) and gives rise to the instantaneous net benefit (4). The optimal transition policy is the solution to

$$V(X_0, K_0) \equiv Max_{\Gamma} \int_0^{\infty} \left[G(q_t^c + q_t^s) - C(q_t^c) - M_s(K_t)q_t^s - R_t\right]e^{-rt}dt \qquad (5)$$

subject to (1)-(2), $q_t^c, q_t^s \geq 0$, $0 \leq R_t \leq \overline{R}$, $X_t \geq 0$, and X_0, K_0 given. In (5), r is the time rate of discount and $\overline{R}$ is an exogenous upper bound on the affordable R&D effort. Together with Eq. (2), this bound implies the upper bound $\overline{K} = \overline{R} / \delta$ on the knowledge state. If δ vanishes or the upper bound on the investment rate R is relaxed, $\overline{K}$ diverges but the results are hardly affected (this case will be further discussed below).

3. CHARACTERIZATION OF THE OPTIMAL POLICY

The complete characterization of the optimal policy requires the specification of the three control processes (q_t^c, q_t^s and R_t) and of the state processes (X_t and K_t) derived thereof. For the problem at hand, this task can be carried out in two steps. First, the optimal supply rates of the primary and backstop resources are determined in much the same way as one would do in a static problem, where the dynamics enter through the resource scarcity rent that is added to the marginal cost of the primary resource and through the current knowledge state. The second step involves the determination of the optimal knowledge and scarcity processes. This second step, it turns out, can be recast as a one-dimensional dynamic optimization problem that admits a most rapid approach solution (Spence and Starrett, 1975) for the knowledge process. Here we characterize the optimal policy, relegating proofs and technical derivations to the appendix.

3.1. Resources supply

The optimal supply rates are determined such that (*a*) the overall supply meets demand, and (*b*) the *effective* marginal cost of the nonrenewable resource equals that of the backstop (which depends on knowledge). The effective marginal cost of the primary resource consists of $M_c(q^c)$ and the scarcity rent $\lambda_t = \lambda_0 e^{rt}$, with λ_0 a nonnegative constant depending on the initial resource stock (see the appendix).

Since at any point of time the optimal supply rates of the primary and backstop resources depend on the knowledge K_t and scarcity rent λ_t levels, we denote these rates by $q^c(K_t,\lambda_t)$ and $q^s(K_t,\lambda_t)$. So long as its stock is not depleted, the primary resource is supplied up to the level where its effective marginal cost just equals the marginal cost of the backstop:

$$M_s(K_t) = M_c(q^c(K_t,\lambda_t)) + \lambda_t \tag{6}$$

(see right panel of Figure 1). Any additional demand beyond this level is supplied by the backstop technology. The overall supply is the rate at which demand (D^{-1}) intersects the minimal unit supply cost. Assuming that the intersection point falls on the flat part of the supply curve, where the latter equals $M_s(K_t)$, the market clearing condition reads

$$q^c(K_t,\lambda_t) + q^s(K_t,\lambda_t) = D(M_s(K_t)) \tag{7}$$

Figure 1

A difficulty with implementing the supply rule (6)-(7) arises when the primary resource supply rate is positive but the resource stock is already depleted. Fortunately, this situation cannot occur under the *optimal* policy. This is so because the optimal K_t and λ_t processes are so chosen that as of the depletion time T^* it is not optimal to use the primary resource. This property stems from the following relations:

$$(a)\ M_s(K^*_{T^*}) = M_c(0) + \lambda^*_0 e^{rT^*} \ \text{ and } \ (b)\ \int_0^{T^*} q^c(K^*_t, \lambda^*_0 e^{rt})dt = X_0\,. \tag{8}$$

Condition (8b) is a restatement of the depletion event at time T^*. Condition (8a) implies that as the depletion time T^* is approached, the optimal rate of the primary resource extraction approaches zero (cf. Eq. 6). Thus, the resource supply does not undergo a discontinuous drop at the depletion time and the backstop technology takes an increasing share prior to depletion. This property stems from our specification of a rate-dependent marginal cost of the primary resource supply, which allows for the simultaneous use of both resources. Hung and Quyen (1993) obtained a similar result by assuming that the marginal cost of the backstop increases with the supply rate.

To implement the supply rule (6)-(7), one needs to determine the optimal knowledge process, to which we now turn.

3.2. *Optimal R&D policy*

Spence and Starrett (1975) defined a Most Rapid Approach Path (MRAP) as the policy that drives the underlying state process to some steady state $\hat{K}$ as rapidly as possible and retains it at that level thereafter. Let

$$K_t^m = (1 - e^{-\delta t})\overline{R} / \delta + K_0 e^{-\delta t} \tag{9}$$

be the knowledge path that departs from K_0 when R&D investment is set at its maximal rate $\overline{R}$ (see Eq. 3). When the depreciation constant δ vanishes, Equation (9) specializes to $K_t^m = \overline{R}t + K_0$. The MRAP policy initiated at $K_0 < \hat{K}$ is given by $Min(K_t^m, \hat{K})$.

In the present case, we find that the optimal R&D policy is to steer the optimal knowledge process K_t^* as rapidly as possible to some target process (to be derived below) and then to continue along the target process to a steady state. To derive the target process, we introduce the function

$$L(K,\lambda) = -M_s'(K)q^s(K,\lambda) - (r+\delta). \tag{10}$$

It turns out (see the appendix) that the target process corresponding to the optimal R&D policy is the root of $L(K,\lambda)$, i.e., the solution $K(\lambda)$ of $L(K(\lambda),\lambda) = 0$ evaluated at the optimal λ-process. The function L is recognized as the evolution function defined and used by Tsur and Zemel (1996, 2001) to identify steady states in a number of dynamic models. While in previous applications L is a function of the state variable only, here, due to the additional state variable X_t and its costate $\lambda_t = \lambda_0 e^{rt}$, the function and its root depend also on time. This is the reason why the MRAP is to the process $K(\lambda)$ rather than to a fixed steady state.

The evolution function and the corresponding root process bear a simple economic interpretation. Increasing the knowledge level by dK reduces the cost of the backstop supply by $-M_s'(K)q^s dK$ but inflicts the cost $(r+\delta)dK$ due to interest payment on the investment and the increased depreciation. At each point of time t, the root $K(\lambda_t)$ of $L(K,\lambda_t)$ represents the optimal balance between these conflicting effects.

These considerations are presented formally in the appendix where it is shown that the R&D problem is equivalent to the problem $Max_{\{R_t\}} \int_0^\infty \vartheta(K_t,\lambda_t)e^{-rt}dt$ subject to (2) and the constraints on R_t, where the effective utility

$$\vartheta(K,\lambda_t) = G(D(M_s(K_t)) - C(q^c(K,\lambda_t)) - \lambda_t q^c(K,\lambda_t) - M_s(K)q^s(K,\lambda_t) - (r+\delta)K,$$

accounts for the net consumer and producer surplus and for the expenses associated with the resource scarcity and the knowledge capital. Since this utility is independent of the control R, it is clear that K must be driven to maximize ϑ as rapidly as possible. Using the supply rule (6)-(7), we obtain $\partial\vartheta/\partial K = L(K,\lambda)$. Thus, aiming at maximizing the objective, we seek the root of $L(K,\lambda)$ over the K-domain in which $L(K,\lambda)$ decreases in K. We maintain that

Assumption 1: The function $L(K,\lambda)$ has a unique root $K(\lambda)$ over the K-domain in which it is decreasing such that $K_0 \le K(\lambda) \le \overline{K} = \overline{R}/\delta$ for any nonnegative λ.

The assumption implies that the initial knowledge level is sufficiently low, and that some R&D activities are worthwhile. Its relaxation would imply corner solutions (e.g. if the root $K(\lambda)$ exceeds $\bar{K}$) or an ambiguity concerning the "correct" root, but otherwise adds no further insight to the analysis (see Tsur and Zemel, 2001, for a discussion of evolution functions with multiple roots).

The process $K(\lambda_t^*)$ driven by the optimal λ_t process is called the *root process*. Its relation to the optimal knowledge process K_t^* is specified as

Proposition 1: The optimal R&D policy is a MRAP with respect to the root process:

$$\text{(a)}\ K_t^* = Min\{K_t^m, K(\lambda_t^*)\}; \quad \text{(b)}\ R_t^* = \begin{cases} \bar{R} & \text{if } K_t^* < K(\lambda_t^*) \\ K'(\lambda_t^*)r\lambda_t^* + \delta K(\lambda_t^*) & \text{if } K_t^* = K(\lambda_t^*) \end{cases} \tag{11}$$

(To avoid trivialities, it is assumed that $\bar{R}$ exceeds the rate implied by Eq. (11) along $K(\lambda_t^*)$.)

In simple terms, Proposition 1 implies the following

Policy Rule: The optimal R&D program should begin immediately at the highest possible rate.

The policy implication of this rule is twofold: First, under Assumption 1 delaying the R&D program cannot be justifiable. Second, the R&D program should be initiated at the highest possible rate. Only later, when the knowledge process reaches the root process, a reduction in the rate of R&D investments is advantageous. Of course, if Assumption 1 is violated and the initial knowledge state K_0 lies above the root process, the optimal policy is to delay R&D activities and let the knowledge depreciate as rapidly as possible and approach the root process from above. Here, however, our interest is focused on backstop technologies that are not yet mature—with knowledge states that are relatively low.

A special case of interest occurs when the upper bound on the affordable R&D effort is removed. In this case, the above Policy Rule implies that the knowledge state is brought immediately to $K(\lambda_0^*)$; the optimal R&D policy, then, reduces to a singular ride along the root process at all times. We further note that the depreciation constant δ shows up in L as an additive component of the discount rate. Thus, setting $\delta = 0$ merely shifts the root process upwards, (see Figure 2), but otherwise does not affect the ensuing optimal R&D policy.

Whether the optimal knowledge process is a MRAP to a fixed state or to a dynamic target process depends on the parameters of the problem (including X_0, K_0, $\overline{R}$, r and δ) and on some benchmark quantities. The full derivation of these quantities and of λ_t^* is presented in the appendix. Here we just give the necessary definitions needed to complete the characterization of the optimal knowledge process.

Let

$$L_\infty(K) = -M_s'(K)D(M_s(K)) - (r+\delta). \tag{12}$$

Comparing with Eq. (10) and noting that $M_s'(K) < 0$ and $D(M_s(K)) \geq q^S(K,\lambda)$, we see that $L_\infty(K)$ bounds $L(K,\lambda)$ from above (Figure 2). Let $\hat{K}$ be the root of $L_\infty(K)$, i.e.,

$$-M_s'(\hat{K})D(M_s(\hat{K})) - (r+\delta) = 0. \tag{13}$$

This root turns out to be the knowledge steady state. Figures 2a-b depict $L_\infty(K)$ and a family of $L(K,\lambda)$ curves corresponding to different λ values. The upper curve corresponds to $L_\infty(K)$ and the lower to $L(K,0)$. Any $L(K,\lambda)$ with $\lambda > 0$ must lie between these two extreme curves.

Another knowledge level of interest is the lowest level of K for which $L(K,0) = L_\infty(K)$, i.e., the state K^S satisfying (cf. Eqs. 10 and 12)

$$M_s(K^S) = M_c(0). \tag{14}$$

K^S is the minimal K-level that renders the backstop cheaper to use than the primary resource even with an infinite stock (hence with zero scarcity rent λ). It can be read off Figure 1 as the knowledge level at which the unit backstop cost ($M_s(K^S)$) equals the cost of the first unit of the resource ($M_c(0)$). In Figures 2a-b, K^S is the level at which $L(K,0)$ and $L_\infty(K)$ coincide. With $\lambda > 0$, $L(K,\lambda)$ and $L_\infty(K)$ coincide at lower K levels.

Figure 2a-b

Figure 2a corresponds to the case $\hat{K} \geq K^S$. It clearly shows that the root process reduces to the singleton $\hat{K}$ in this case. In view of Proposition 1, therefore, we conclude that if $\hat{K} \geq K^S$, the optimal R&D policy is a MRAP to $\hat{K}$. If $\hat{K} < K^S$, the root process changes over time (Figure 2b). Nevertheless, a MRAP to $\hat{K}$ may still be optimal. This happens when the initial nonrenewable stock X_0 does not exceed the benchmark quantity Q^m, defined as the amount of primary resource stock needed to carry on the MRAP policy $K_t^* = Min\{K_t^m, \hat{K}\}$ from the initial time until $\hat{K}$ is reached (see Eq. A21 in the appendix). This is so because a low initial stock (below Q^m) gives rise to a high scarcity rent, which in turn implies a high root process that will not be crossed by K_t^m prior to arrival at $\hat{K}$. We summarize the conditions under which the optimal policy is a MRAP to $\hat{K}$, i.e., $K_t^* = Min\{K_t^m, \hat{K}\}$, in

Proposition 2: If either (i) $\hat{K} \geq K^S$ or (ii) $\hat{K} < K^S$ and $X_0 \leq Q^m$, then $K_t^* = Min\{K_t^m, \hat{K}\}$.

If neither (i) nor (ii) holds, the optimal R&D policy is a MRAP to the root process:

Proposition 3: If $\hat{K} < K^S$ and $X_0 > Q^m$, then $K_t^* = K_t^m$ until some date τ satisfying $K_\tau^m = K(\lambda_\tau^*)$, following which $K_t^* = K(\lambda_t^*)$ until the depletion date T^*. At the depletion time, $K(\lambda_t^*)$ and the optimal process K_t^* arrive at the steady state $\hat{K}$ and settle at this state.

We observe in Fig. 2a that $K(\lambda)$ increases with λ. Since the latter increases exponentially with time, it follows that the root process also increases with time, until the steady state is reached. Thus, the optimal process described in Proposition 3 contains three distinct phases: (i) rapid increase along the MRAP; (ii) gradual increase along the root process; and (iii) resting at the steady state. The switching time τ between phases (i) and (ii), the depletion date T^*, separating phases (ii) and (iii), and λ_0^* are derived in the appendix.

Under the conditions of Proposition 3, the balance between the backstop technology cost reduction and knowledge depreciation is different prior to depletion, while extraction is still feasible (as represented by the root $K(\lambda)$ of $L(K,\lambda)$), from that after depletion (as represented by the root $\hat{K}$ of $L_\infty(K)$). If a large initial stock prevents early depletion, investing in R&D at the maximum possible rate entails knowledge depreciation in excess of what is justified by the backstop technology cost reduction, hence cannot be optimal. The investment rate, therefore, is decreased prior to arrival at the steady state. It is of interest to note that even in this case the slowdown in R&D investment occurs only at the final, singular part of the knowledge process.

Given K_t^* and $\lambda_t^* = \lambda_0^* e^{rt}$, the optimal primary/backstop supply rates are given by (6)-(7) (see also Figure 1), completing the characterization of the optimal policy.

4. CONCLUDING COMMENTS

The received literature on the development of backstop technologies to scarce resources considers technical change processes that come about in the form of major breakthroughs or in discrete steps. Here we consider smooth and gradual technical change processes in which the cost of an existing technology is continuously reduced as a result of R&D efforts that increase its knowledge base. The two approaches, it turns out, entail markedly different R&D policies.

We find that if R&D is at all worthwhile, it should be initiated at the maximal affordable rate with no delay and possibly be decreased later on as the knowledge process reaches a (derived) target process. Thus, the model advocates substantial early engagement in R&D programs that should precede, rather than follow, future increases in the price of the primary resource

The particulars of the Most Rapid Approach Path derived here for the knowledge process are technically related to the assumed linearity of the learning process, as the knowledge stock is the accumulation of R&D expenditures (possibly with a depreciation term). This observation calls for a few comments. First, while a simplification, the economic intuition supporting large and early R&D efforts is associated with the fact that the benefits resulting from these efforts are immediate and need not await some (known or uncertain) date of a major technological breakthrough. Therefore, delaying the R&D efforts is suboptimal. Second, the overall effect of R&D on the cost of backstop supply is far from linear, as the cost decreases with the stock of knowledge in a nonlinear fashion. Thus, the diminishing returns associated with the knowledge process are present also in this model, and are manifested via the evolution function L and the ensuing root process. Finally, the current structure can be viewed as an approximation to a nonlinear learning process of the form $\dot{K} = y(R) - \delta K$ with the general increasing and concave learning function $y(R)$ replaced by the piecewise linear function $y(R) = R$ or $y(R) = \bar{R}$ as R falls short or exceeds $\bar{R}$. Evidently, with general concave learning the initial R&D effort may not be determined by an exogenous upper bound but rather by the curvature of the learning function, but the recommendation of early engagement in R&D activities is preserved. A detailed investigation of this case is left for future research.

Considerations of time-dependent demand (Tsur and Zemel, 1998) and renewable primary resources (Tsur and Zemel 2000b) suggest that these findings hold in more general situations. Another extension involves externalities associated with the use of the primary resource, e.g., polluting emissions due to the use of fossil energy. This case is considered in Tsur and Zemel (2000a), where the externality increases the effective cost of the primary resource, but the MRAP nature of the optimal R&D policy remains the same.

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APPENDIX: DERIVATION OF THE OPTIMAL TRANSITION POLICY

Preliminaries: The resource stock depletion date T, when finite, divides the planning period into two distinct sub-periods: the *pre-depletion* period in which it is possible to supply from both the primary and the backstop sources, and the *post-depletion* period when only the backstop is available. With T a decision variable, the optimization problem (5) is recast as

$$V(X_0, K_0) = Max_{\{\Gamma, T\}} \int_0^T \left[G(q_t^c + q_t^s) - C(q_t^c) - M_s(K_t) q_t^s - R_t \right] e^{-rt} dt + e^{-rT} V(0, K_T) \quad \text{(A1)}$$

subject to (1)–(2), $q_t^c, q_t^s \geq 0$, $0 \leq R_t \leq \overline{R}$, $X_T \geq 0$ (equality holding when T is finite), and given X_0 and K_0. It is recalled that $\Gamma = \{ q_t^c, q_t^s, R_t, t \geq 0\}$ and $G(q) = \int_0^q D^{-1}(z) dz$. The post-depletion value function $V(0,K)$ is expressed as

$$V(0, K) \equiv V^S(K) = Max_{\{q_t^s, R_t\}} \int_0^\infty [G(q_t^s) - M_s(K_t) q_t^s - R_t] e^{-rt} dt \quad \text{(A2)}$$

subject to (2), $0 \leq R_t \leq \overline{R}$, $q_t^s \geq 0$ and $K_0 = K$ (the initial time for the post-depletion problem is reset to $t = 0$). It is easy to verify that the optimal post–depletion rate of backstop supply equals $q_t^s = D(M_s(K_t))$, hence the post–depletion problem can be recast as

$$V^S(K) = Max_{\{R_t\}} \int_0^\infty [G(D(M_s(K_t))) - M_s(K_t) D(M_s(K_t)) - R_t] e^{-rt} dt \quad \text{(A3)}$$

subject to (2), $0 \leq R_t \leq \overline{R}$ and $K_0 = K$.

The current-value hamiltonian for (A1) is of the form

$H_t = G(q_t^c + q_t^s) - C(q_t^c) - M_s(K_t) q_t^s - R_t - \lambda_t q_t^c + \gamma_t (R_t - \delta K_t)$, where λ_t and γ_t are the current-value costate variables corresponding to X_t and K_t, respectively. Incorporating the Lagrange multipliers α^c and α^s associated with the nonnegativity of the supply rates as well as α^0 and

α^R associated with the constraints on R_t, the Lagrangian

$\Im_t = H_t + \alpha_t^c q_t^c + \alpha_t^s q_t^s + \alpha^0 R_t + \alpha^R(\overline{R} - R_t)$ is obtained. The necessary conditions include (see, e.g., Leonard and Long, 1992; all quantities are evaluated at their optimal values):

(a) $\partial\Im_t/\partial q_t^c = 0$ and $\partial\Im_t/\partial q_t^s = 0 \Rightarrow$

$$D^{-1}(q_t^c + q_t^s) - M_c(q_t^c) - \lambda_t + \alpha_t^c = 0 \quad \text{and} \quad D^{-1}(q_t^c + q_t^s) - M_s(K_t) + \alpha_t^s = 0 \tag{A4}$$

with the complimentary slackness conditions $\alpha_t^c q_t^c = \alpha_t^s q_t^s = 0$.

(b) $\dot{\lambda}_t - r\lambda_t = -\partial H_t / \partial X_t = 0 \Rightarrow$

$$\lambda_t = \lambda_0 e^{rt}. \tag{A5}$$

(c) The transversality conditions associated with $X_T \geq 0$ read $\lambda_0^* X_T^* = 0$ and $\lambda_0 \geq 0$, equality holding if the stock is never depleted. From (A4)-(A5) we see that both

$$M_s(K_t) = M_c(q_t^c) + \lambda_0 e^{rt} - \alpha_t^c \tag{A6}$$

and

$$q_t^c + q_t^s = D(M_s(K_t)) \tag{A7}$$

hold along the optimal plan whenever q_t^s is positive (so that α_t^s vanishes), verifying the supply rule (6)-(7) for $q^c(K_t,\lambda_t)$ and $q^s(K_t,\lambda_t)$, as displayed in Figure 1.

(d) $\dot{\gamma} - r\gamma = -\partial H / \partial K$ gives

$$\dot{\gamma} = M_s'(K) q^s(K,\lambda) + \gamma(r + \delta) \tag{A8}$$

and the transversality condition associated with the free value of K_T reads

$$\gamma_T = \partial V(0,K_T)/\partial K = V^{s\prime}(K_T). \tag{A9}$$

Thus, the costate variable γ_t evolves smoothly as the pre-depletion problem (A1) turns into the post-depletion problem (A3) at time T (note that $V^{S\prime}(K_T)$ equals the initial value of the costate variable γ of the post-depletion problem for which K_T is the initial state).

(e) Maximizing the Lagrangians of the pre- and post-depletion problems with respect to R_t

reveals that either $R_t = 0$ or $R_t = \overline{R}$ whenever $\gamma_t \neq 1$. It follows that the process R_t can undergo a discontinuity only at the singular value $\gamma_t = 1$, and the quantity $R_t(\gamma_t - 1)$ is continuous in time.

(*f*) The transversality condition associated with the free choice of T is

$$H_T = rV^S(K_T). \tag{A10}$$

Proof of the continuity property (8a): Let the subscripts "–" and "+" denote, respectively, the pre- and post-depletion limits $t \to T$ from below and $t \to T$ from above. The above-listed transversality conditions of the pre-depletion problem (A1) correspond to the former limit, hence the subscripts "–" and "T" for these conditions bear the same meaning, while the "+" subscript is attached to the initial values for the optimal processes of the post- depletion problem (A3).

In view of (A6), condition (8a) follows if $q_-^c = 0$ and $\alpha_-^c = 0$, so that the stock will *not* be depleted *before* the effective cost of the primary resource is high enough to exclude its supply. To show this, recall from (*d*) and (*e*) above that $\gamma_- = \gamma_+ = V^{s\prime}(K_T)$ and $R_-(\gamma_- - 1) = R_+(\gamma_+ - 1)$. Moreover, the knowledge process is also continuous at the depletion date T, hence the notation K_T bears no risk of confusion.

Using the expression in (A5) for λ_t, we obtain

$$H_- \equiv H_T = G(q_-^c + q_-^s) - C(q_-^c) - \lambda_0 e^{rT} q_-^c - M_s(K_T) q_-^s + R_-(\gamma_- - 1) - \gamma_- \delta K_T \tag{A11}$$

According to (A10), the right-hand side of (A11) should equal $rV^S(K_T)$.

The Dynamic Programming (Bellman) equation for the autonomous post depletion problem (A3) reads (Kamien and Schwartz, 1981, p. 242):

$$rV^s(K_T) = G(D(M_s(K_T))) - M_s(K_T)D(M_s(K_T)) - R_+ + V^{s\prime}(K_T)[R_+ - \delta K_T] \tag{A12}$$

$$= G(D(M_s(K_T))) - M_s(K_T)D(M_s(K_T)) + R_+(\gamma_+-1) - \gamma_+\delta K_T$$

where R_+ is the optimal initial investment rate for the post-depletion problem. Recalling the continuity of γ_t, K_t and $R_t(\gamma_t-1)$ at $t = T$, we conclude from (A10–A12) that $G(q_-^c + q_-^s) - C(q_-^c) - \lambda_0 e^{rT} q_-^c = G(D(M_S(K_T))) - M_S(K_T)[D(M_S(K_T)) - q_-^s]$. Invoking (A6), (A7) and $\alpha_-^c q_-^c = 0$, this result reduces to

$$C(q_-^c) - M_c(q_-^c)q_-^c = 0. \tag{A13}$$

Now, $M_c(q^c) \equiv C'(q^c)$, C is strictly convex with $C(0) = 0$, hence the function $C(q^c) - M_c(q^c)q^c$ decreases with q^c and vanishes at $q^c = 0$. Therefore, (A13) implies that $q_-^c = 0$. The supply of the primary resource, therefore, vanishes upon depletion in a continuous manner. Just before T, however, q_t^c cannot vanish (for otherwise depletion cannot occur), hence the Lagrange multiplier α_t^c must vanish at the final stage of the pre-depletion problem. Taking the limit $t \to T$ from below of (A6) (with $\alpha_t^c = 0$) yields (8a), while (8b) holds trivially when the stock is depleted. •

Proof of Proposition 1: For a given scarcity process λ_t, consider the optimization problem:

$$v(K_0) \equiv Max_{\{R_t\}} \int_0^\infty \tilde{\vartheta}(K_t, R_t, \lambda_t) e^{-rt} dt\,. \tag{A14}$$

subject to $\dot{K}_t = R_t - \delta K_t$; $0 \le R_t \le \overline{R}$ and K_0 given, where

$$\tilde{\vartheta}(K_t, R_t, \lambda_t) = G(D(M_s(K_t))) - C(q^c(K_t, \lambda_t)) - \lambda_t q^c(K_t, \lambda_t) - M_s(K_t) q^s(K_t, \lambda_t) - R_t$$

and $q^c(K_t,\lambda_t)$ and $q^s(K_t,\lambda_t)$ are the optimal supply rates specified in (6)-(7). It can be verified that the necessary conditions corresponding to (A14) coincide with the necessary conditions (*d*)-(*e*) associated with γ_t and R_t of problem (A1) (except for the transversality condition that must correspond to $t = \infty$ rather than to $t = T$; see remark (*ii*) below). Following Spence and

Starrett (1975), we use (2) to remove R from $\tilde{\vartheta}$. Integrating the term involving $\dot{K}$ by parts, we find that (A14) is equivalent to

$$v(K_0) = K_0 + Max_{\{R_t\}} \int_0^\infty \vartheta(K_t, \lambda_t) e^{-rt} dt \tag{A15}$$

subject to $\dot{K}_t = R_t - \delta K_t$; $0 \le R_t \le \overline{R}$ and K_0 given, where

$$\vartheta(K, \lambda_t) = G(D(M_s(K_t)) - C(q^c(K, \lambda_t)) - \lambda_t q^c(K, \lambda_t) - M_s(K) q^s(K, \lambda_t) - (r + \delta)K$$

Taking the derivative of $\vartheta(K,\lambda)$ with respect to K and using (A6)-(A7), we obtain $\partial\vartheta/\partial K = -M_s'(K)q^s(K,\lambda) - (r+\delta) \equiv L(K,\lambda)$, as specified in (10). Thus, a root $K(\lambda_t)$ of $L(K,\lambda)$ (i.e. a solution to $L(K(\lambda_t),\lambda_t) = 0$) in the region where L decreases in K, maximizes $\vartheta(K,\lambda_t)$ at any time t. According to Assumption 1, a unique feasible maximum exists for every positive λ, hence the root process $K(\lambda_t)$ is well defined.

Since the equivalent utility ϑ in (A15) is independent of R, the optimal policy is to approach the maximal ϑ as rapidly as possible. Now, the time dependence of λ_t (see A5) induces a corresponding time dependence on ϑ and on the root process. Therefore, the optimization problem (A15) is not autonomous. Nevertheless, the argument of Spence and Starrett (1975, footnote, p. 394) can be invoked to establish that the optimal policy is a MRAP to the root process $K(\lambda_t)$. Once the root process has been reached, ϑ must be maintained at its maximum by tuning R_t so as to ensure that $K_t = K(\lambda_t)$ for the rest of the process, i.e., $R_t - \delta K_t = \dot{K}_t = dK(\lambda_t)/dt = K'(\lambda_t) r \lambda_t$, as specified in (11). •

Remarks: (*i*) From the continuity property established above, it follows that (A15) and (A3) yield the same solution for the post depletion period $t > T$ and there is no need to solve (A3) independently.

(*ii*) (A8) and the transversality condition $e^{-rt}\gamma_t \to 0$ as $t \to \infty$, associated with the free value of K_∞ in (A14), give $\gamma_t = e^{(r+\delta)t}\int_t^\infty -M_s'(K_\tau)q^s(K_\tau^{\ *},\lambda_\tau)e^{-(r+\delta)\tau}d\tau$. Using (10) and the identity $1 = e^{(r+\delta)t}\int_t^\infty (r+\delta)e^{-(r+\delta)\tau}d\tau$, we find that $\gamma_t - 1 = e^{(r+\delta)t}\int_t^\infty L(K_\tau\lambda_\tau)e^{-(r+\delta)\tau}d\tau$. Thus, as soon as the optimal K-process reaches the root process and evolves together with it, the integral on the right hand side above vanishes, retaining the singular value $\gamma_t = 1$ for the rest of the process. Indeed, (A8) and (10) imply that $\dot{\gamma} = -L(K,\lambda)$ at the singular value.

The R&D and scarcity rent processes:

We begin with the case $\hat{K} \geq K^S$ (Figure 2a):

Proof of Proposition 2(i): Any knowledge level above K^S excludes the use of the primary resource hence $q^c(\hat{K},\lambda) = 0$ and $L(\hat{K},\lambda) = L_\infty(\hat{K}) = 0$ for any λ. Thus, $K(\lambda) = \hat{K}$ identically for all λ, implying that the root process reduces to the singleton $\hat{K}$ and, according to Proposition 1, the optimal R&D policy is the MRAP $K_t^* = Min\{K_t^m, \hat{K}\}$ •

Before turning to Case (ii) of Proposition 2 we characterize the optimal scarcity rent process under the present case of $\hat{K} \geq K^S$. The optimal scarcity process is of the form $\lambda_t^* = \lambda_0^* e^{rt}$ (see A5) and its characterization requires the parameter λ_0^*, which depends on the initial reserves in the following way. Let T^S denote the time when $K_t^m = K^S$. Using (9), and recalling $\overline{K} = \overline{R}/\delta$, we find

$$T^S = \log[(\overline{K} - K_0)/(\overline{K} - K^S)]/\delta \tag{A16}$$

(when $\delta = 0$, $K_t^m = K_0 + \overline{R}t$ and $T^S = (K^S - K_0)/\overline{R}$). Let Q^0 be the total amount of the resource consumed under the MRAP $K_t^* = K_t^m$ with an unbounded initial stock and a vanishing scarcity rent:

$$Q^0 = \int_0^\infty q^c(K_t^m, 0)dt = \int_0^{T^S} q^c(K_t^m, 0)dt\,. \tag{A17}$$

(Recall that $q^c(K,0) = 0$ for $K \geq K^S$ regardless of the remaining reserves, hence $q^c(K_t^m,0) = 0$ for $t \geq T^S$.) Suppose that $X_0 \geq Q^0$ and $\lambda_0^* > 0$. Then, since $q^c(K,\lambda)$ decreases in λ, $\int_0^\infty q^c(K_t^m, \lambda_t^*)dt < \int_0^\infty q^c(K_t^m, 0)dt = Q^0 \leq X_0$ so that the stock is never depleted, violating the transversality condition (*c*), hence $\lambda_0^* = 0$. It follows that if $Q^0 \leq X_0$ (i.e., if the initial stock X_0 is large enough to support the primary resource exploitation plan $\{q^c(K_t^m,0),\ t\geq 0\}$), then the scarcity rent must vanish.

We now show that an initial stock below Q^0 implies depletion and a positive λ_0^*. Suppose that $X_0 < Q^0$ but $\lambda_0^* = 0$. Then $X_{T^S} = X_0 - Q^0 < 0$, violating $X_t \geq 0$. Thus, $\lambda_0^* > 0$ and, in view of the transversality condition (*c*), the stock must be depleted and the parameters λ_0^* and T^* are found by solving equations (8a-b). To sum, if $\hat{K} \geq K^S$, then:

(1) The optimal R&D policy is the MRAP $K_t^* = Min\{K_t^m, \hat{K}\}$.

(2) If $X_0 \geq Q^0$ then λ_t^* vanishes identically for all t.

(3) If $X_0 < Q^0$ then $\lambda_t^* = \lambda_0^* e^{rt} > 0$, the resource reserves will be depleted at a finite date T^*, and λ_0^* and T^* are found by solving equations (8a-b).

In (1) above case (i) of Proposition 2 is restated; (2) and (3) reveal an obvious dependence of λ_0^* on the initial stock.

We turn now to the case $\hat{K} < K^S$ (Figure 2b). Since the initial knowledge level K_0 lies below $K(\lambda)$ for any $\lambda \geq 0$, Proposition 1 implies that the optimal process K_t^* evolves initially along K_t^m. If K_t^m overtakes the root process $K(\lambda_t^*)$ before the latter reaches $\hat{K}$, then K_t^* switches to $K(\lambda_t^*)$ and continues with it as a singular process until they arrive at $\hat{K}$.

Otherwise, the optimal process evolves as a MRAP along K_t^m all the way to $\hat{K}$.

Whether or not the processes K_t^m and $K(\lambda_t^*)$ cross before they reach $\hat{K}$ depends on the initial scarcity rent λ_0^*. For example, when $\lambda_0^* = 0$, $K(\lambda_t^*)$ is fixed at $K(0)$ and will surely be overtaken by K_t^m; at the other extreme, for large enough λ_0^*, $K(\lambda_t^*) = \hat{K}$ already at $t = 0$.

Let $\hat{T}$ be the date at which K_t^m reaches $\hat{K}$:

$$\hat{T} = \log[(\overline{K} - K_0)/(\overline{K} - \hat{K})]/\delta\,. \tag{A18}$$

Define

$$\hat{\lambda} = M_S(\hat{K}) - M_S(K^S)\,. \tag{A19}$$

Our assumption that $\hat{K} < K^S$ ensures that $\hat{\lambda} > 0$. Using these quantities we define

$$\lambda_0^m = \hat{\lambda} e^{-r\hat{T}} \quad \text{and} \quad \lambda_t^m = \lambda_0^m e^{rt} \tag{A20}$$

and establish the following criterion for the optimal process K_t^* to obtain a singular branch along the root process:

Lemma: (*a*) If $\lambda_0^* \geq \lambda_0^m$, then $K_t^* = Min(K_t^m, \hat{K})$ is a MRAP to steady state $\hat{K}$.

(*b*) If $\lambda_0^* < \lambda_0^m$, then $K_t^* = Min(K_t^m, K(\lambda_0^* e^{rt}))$ is a MRAP to the root process.

Proof: (*a*) Equations (14), (A19) and (A20) imply that $\lambda_{\hat{T}}^m = \hat{\lambda}$ and $M_c(0) + \lambda_{\hat{T}}^m = M_s(\hat{K})$, and the optimal supply rule (6)-(7) reads $q^c(\hat{K}, \lambda_{\hat{T}}^m) = 0$ and $q^s(\hat{K}, \lambda_{\hat{T}}^m) = D(M_S(\hat{K}))$. Thus, $L(\hat{K}, \lambda_{\hat{T}}^m) = L_\infty(\hat{K}) = 0$ and $K(\lambda_{\hat{T}}^m) = \hat{K} = K_{\hat{T}}^m$ (the latter equality follows from the definition of $\hat{T}$ in (A18)). It follows that the root process $K(\lambda_t^m)$ and K_t^m first cross at the date $\hat{T}$ and $K_t^m < K(\lambda_t^m)$ for $t < \hat{T}$. (The two processes cannot cross twice since the MRAP K_t^m is faster than $K(\lambda_t^m)$.) $\lambda_0^* \geq \lambda_0^m$ entails $K(\lambda_t^*) \geq K(\lambda_t^m)$, hence $K_t^m < K(\lambda_t^*)$ for all $t < \hat{T}$, implying that K_t^* is the MRAP $Min(K_t^m, \hat{K})$ to $\hat{K}$.

(*b*) When $\lambda_0^* < \lambda_0^m$, $K(\lambda^*_{\hat{T}}) < K^m_{\hat{T}}$ hence the processes K_t^m and $K(\lambda_t^*)$ cross at some date $\tau < \hat{T}$, at which time, according to Proposition 1, the optimal process K_t^* switches from K_t^m to the root process $K(\lambda_t^*)$ and increases along with it to the steady state $\hat{K}$. •

However, λ_0^* is not known apriori and the above criterion cannot be readily applied. For an equivalent criterion, as given in Proposition 2(ii), we consider the benchmark stock

$$Q^m = \int_0^\infty q^c(K_t^m, \lambda_t^m)dt \tag{A21}$$

needed to carry out the primary resource exploitation plan with K_t^m and λ_t^m as the knowledge and scarcity rent processes. It is verified, using (14) and (A18)-(A20), that the integrand of (A21) vanishes for all $t > \hat{T}$.

Proof of Proposition 2(ii): Given $Q^m \geq X_0$ we show that $\lambda_0^* \geq \lambda_0^m$. Suppose otherwise, that $\lambda_0^* < \lambda_0^m$. Since the process K_t^m is the upper bound of all feasible *K*-processes, $K_t^* \leq K_t^m$ for all *t*. Moreover, $q^c(K,\lambda)$ decreases in both arguments, and the amount of the primary resource required to sustain the policy with K_t^* and $\lambda_t^* \equiv \lambda_0^* e^{rt}$ is

$\int_0^\infty q^c(K_t^*, \lambda_t^*)dt > \int_0^\infty q^c(K_t^m, \lambda_t^m)dt = Q^m \geq X_0$, hence the policy (K_t^*, λ_t^*) is not feasible. Thus, $\lambda_0^* \geq \lambda_0^m$ and Proposition 2(ii) follows from Lemma (*a*). •

Proof of Proposition 3: Given $Q^m < X_0$, we show that $\lambda_0^* < \lambda_0^m$. Suppose otherwise, that $\lambda_0^* \geq \lambda_0^m$. Then, by Lemma (*a*), $K_t^* = K_t^m$. Since q^c decreases with λ,

$\int_0^\infty q^c(K_t^*, \lambda_t^*)dt \leq \int_0^\infty q^c(K_t^m, \lambda_t^m)dt = Q^m < X_0$, implying that the stock is never depleted and violating the transversality condition (*c*). Thus, $\lambda_0^* < \lambda_0^m$ and the Proposition follows from Lemma (*b*). •

We summarize the case $\hat{K} < K^S$:

(1) If $X_0 \leq Q^m$, then the optimal R&D process is a MRAP to $\hat{K}$, the stock will be depleted at a finite date T^*, and the parameters $\lambda_0{}^*$ and T^* are found by solving equations (8a-b).

(2) If $X_0 > Q^m$, then the optimal process $K_t{}^*$ begins as the MRAP and switches at some date $\tau < \hat{T}$ from $K_t{}^m$ to the root process $K(\lambda_t{}^*)$, following an increasing singular branch. The parameters $\lambda_0{}^*$, T^* and τ are obtained by solving simultaneously equations (8a-b) and

$$\tau = \log[(\overline{K} - K_0)/(\overline{K} - K(\lambda_\tau^*))]/\delta\,, \qquad \text{(A22)}$$

which defines τ as the time at which the process $K_t{}^m$ crosses the root process $K(\lambda_t{}^*)$.

Figure 1: *Right panel*: Resource demand and supplies at time t, given K_t and λ_t. The area ABCD represents the sum of consumer and producer surpluses.
Left panel: Marginal cost of the backstop resource as a function of knowledge.

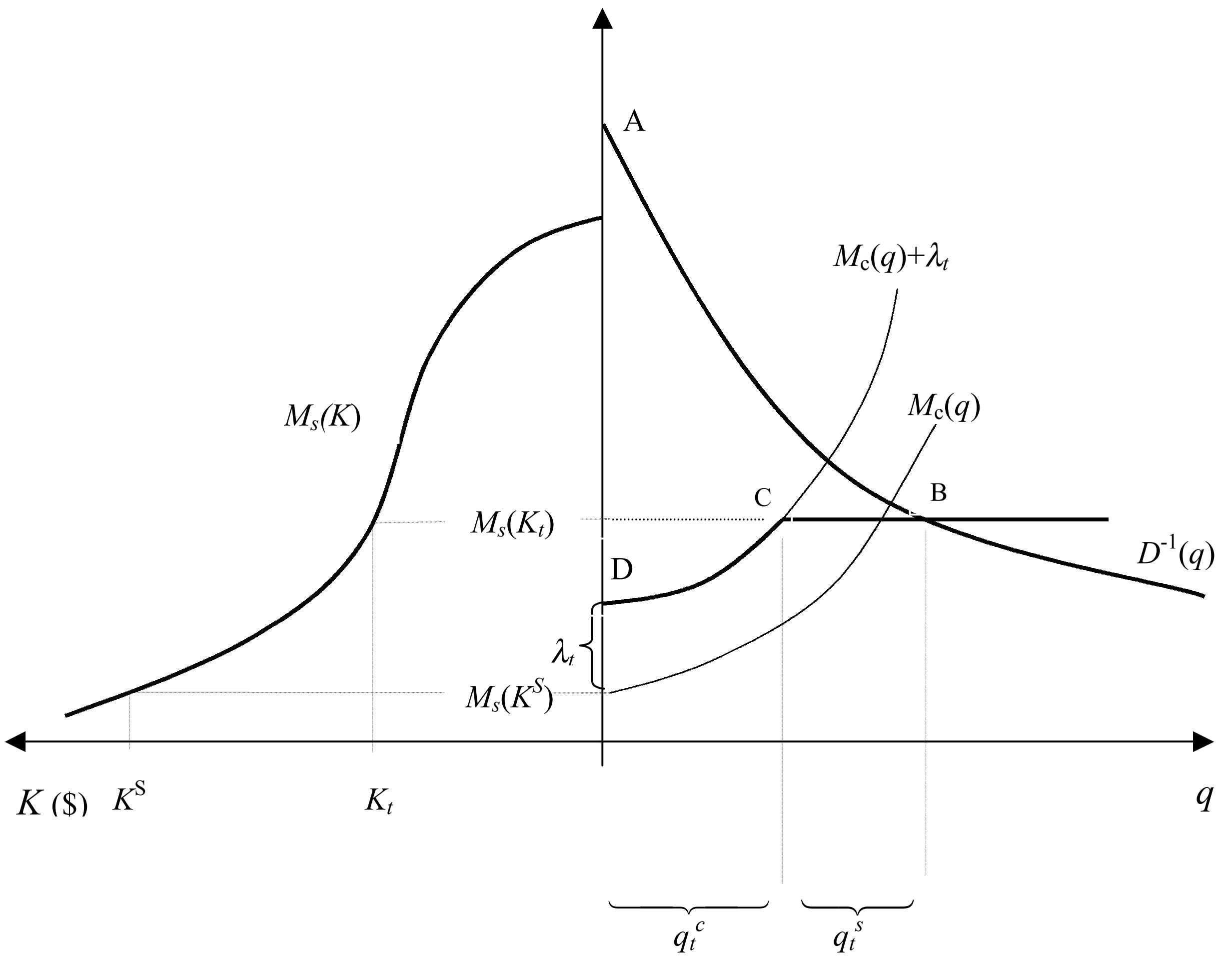

Figure 2a: The evolution functions $L(K,\lambda)$ (Equation 10) and $L_\infty(K)$ (Equation 12) vs. the knowledge level K when $K^S < \hat{K}$. K^S is the critical knowledge level in which $M_s(K^S) = M_c(0)$ and is also the intersection of $L_\infty(K)$ and $L(K,0)$.

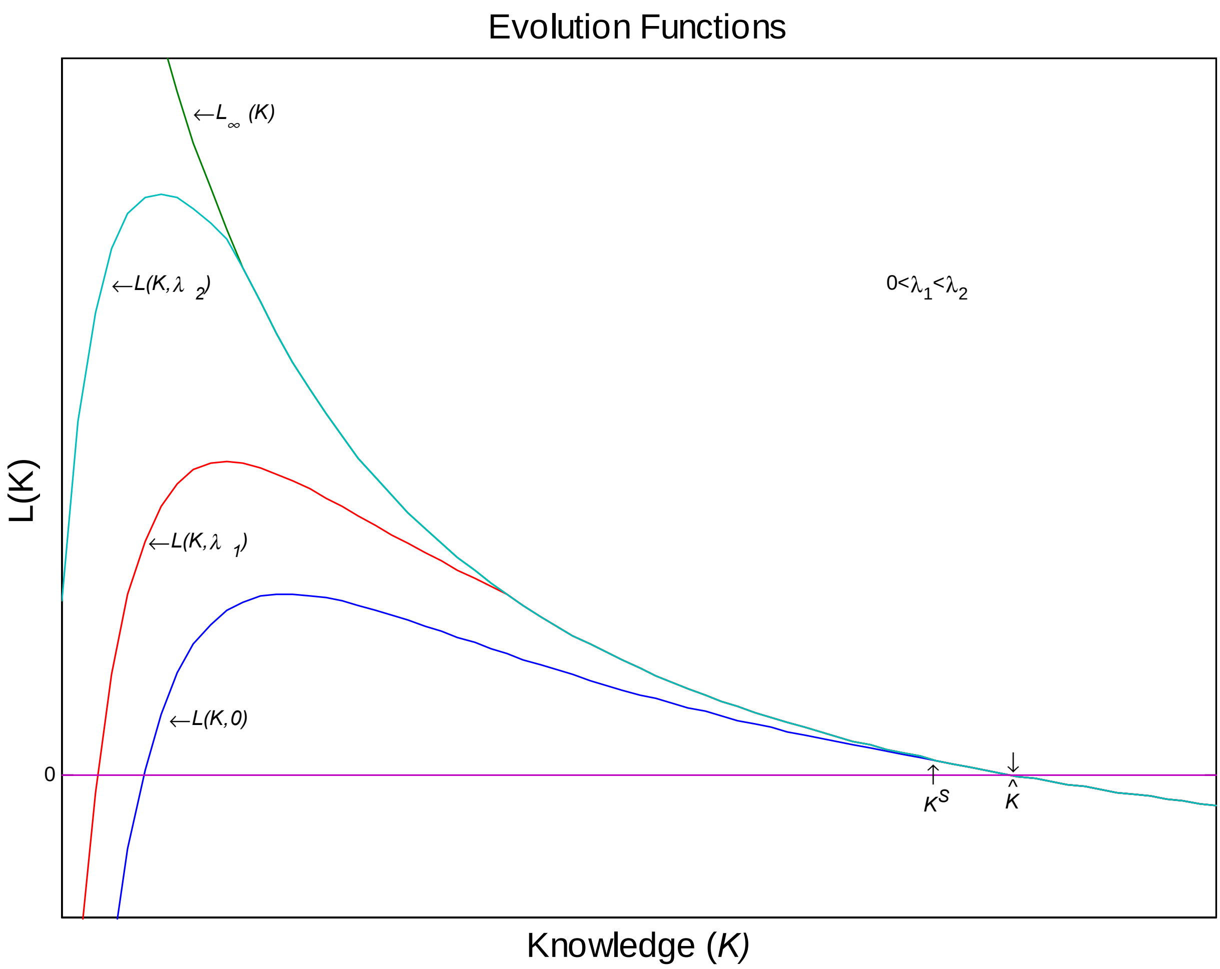

Figure 2b: The evolution functions $L(K,\lambda)$ (Equation 10) and $L_\infty(K)$ (Equation 12) vs. the knowledge level K when $K^S > \hat{K}$. K^S is the critical knowledge level in which $M_s(K^S) = M_c(0)$ and is also the intersection of $L_\infty(K)$ and $L(K,0)$. Both $L_\infty(K)$ and $L(K,\hat{\lambda})$ vanish at $\hat{K}$.